

# Late-time power-law stages of cosmological evolution in teleparallel gravity with nonminimal coupling

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## Abstract

We investigate evolution of the Universe at late-time stages in models of teleparallel gravity with the power-law nonminimal coupling and the decreasing power-law potential of a scalar field  $\phi$ . New asymptotic solutions are found analytically for considered models in vacuum and with a perfect fluid. Applying numerical integration we show that the cosmological evolution leads to these solutions for some region of initial data and these asymptotic regimes are stable in time. The physical sense of obtained results is discussed.

## 1 Introduction

Final stages of cosmological evolution are studied widely as in General Relativity (GR) as in its modifications. It is interesting and urgent especially due to the need of an explanation of observation data [1] (see also the review [2]) indicating the late-time accelerated expansion of the Universe. A realistic models should describe observations and, moreover, do not suffer of shortcomings. There are such problems, as “fine-tuning” of initial conditions for a realization of the late-time cosmic acceleration in models of  $\Lambda$ CDM [3] and quintessence [4] based on GR, future singularities like “Big rip”, “sudden” and others (see, for example, [5]). Cosmological evolution leads to “Big Rip” in some models of modified gravity [6] and in its generalizations [7].

There is the alternative formulation of GR — the teleparallel gravity (“Teleparallel Equivalent of General Relativity”, TEGR [8]). It based, firstly, on Einstein’s idea of absolute parallelism [9] that is on using of a field of orthonormal bases — tetrads — for tangent space-times and, secondly, TEGR applies the Weitzenböck connection [10] instead Levi-Civita one, that leads to zero curvature and non-zero torsion. The TEGR Lagrangian contains the torsion scalar  $T$ , and equations of motion of this theory coincide exactly with those of GR [11, 12, 13]. However, modifications of teleparallel gravity (for example,  $f(T)$  theory [14, 15, 16], scalar-torsion gravity [17, 18, 19, 20]) are not equivalents of analogical modifications of GR, their difference is given by a term with the divergence of torsion in the Lagrangian (see [21]). It gives rise to different field equations and, consequently, new cosmological dynamics. Therefore, it is interesting to investigate modifications of teleparallel gravity. The scenario with the late-time cosmic acceleration were already found, for example, in teleparallel gravity with nonminimal coupling of the form  $\xi T \phi^2$  [17, 18, 19].

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In our recent works [22], [23] stable asymptotic solutions (attractors of corresponding dynamical systems) have been obtained in models of teleparallel gravity with nonminimal coupling of the form  $\xi T\phi^N$  for  $N = 2$  and the potential of the scalar field  $V(\phi) = V_0\phi^n$ . Those methods of dynamical system theory did not allow us to investigate cases of  $N > 2$ ,  $\xi > 0$  and  $n < 0$ . Therefore, in the present work we shall find out final stages of the Universe evolution in such models which are not studied earlier. Units  $\hbar = c = 1$  will be used.

The structure of this paper is the following. In Sect. 2 we present briefly foundations of teleparallel gravity and write basic equations for considered models of scalar-torsion gravity. In Sect. 3 obtained analytical and numerical results are described. Then those are discussed in Sect. 4.

## 2 Basic equations

Let us present shortly foundations of teleparallel gravity and write main equations for its modification with a nonminimally coupled scalar field.

In teleparallel gravity dynamical variables are four linearly independent vectors — the tetrad  $\mathbf{e}_A(x^\mu) = e^\mu_A \partial_\mu$ , where Greek indices are space-time ones, capital Latin indices are tangent space-time those. A tetrad forms orthonormal basis for the tangent space at the each point of space-time. Then the metric tensor is

$$g_{\mu\nu} = \eta_{AB} e^\mu_A e^\nu_B, \quad (1)$$

where  $\eta_{AB} = \text{diag}(1, -1, -1, -1)$  — the Minkowski metric tensor,  $\eta_{AB} = \mathbf{e}_A \cdot \mathbf{e}_B$ . The determinant consisting of tetrad components  $e^\mu_A$  is  $e = \det(e^\mu_A) = \sqrt{-g}$ .

The Weyssenhoff connection is used in teleparallel gravity

$$\overset{\mathbf{w}}{\Gamma}_{\mu\nu}^\lambda \equiv e^\lambda_A \partial_\mu e^\nu_A, \quad (2)$$

which leads to the curvature scalar  $\overset{\mathbf{w}}{R} = 0$ , while the torsion tensor and the torsion scalar are

$$T_{\mu\nu}^\lambda \equiv \overset{\mathbf{w}}{\Gamma}_{\nu\mu}^\lambda - \overset{\mathbf{w}}{\Gamma}_{\mu\nu}^\lambda = e^\lambda_A (\partial_\mu e^\nu_A - \partial_\nu e^\mu_A), \quad (3)$$

$$T \equiv \frac{1}{4} T^{\rho\mu\nu} T_{\rho\mu\nu} + \frac{1}{2} T^{\rho\mu\nu} T_{\nu\rho\mu} - T^\rho_{\rho\mu} T^\mu_{\nu\nu}. \quad (4)$$

The TEGR action have the form  $S = \frac{1}{16\pi G} \int T e d^4x$ . If we add to it the nonminimally coupled scalar field with the potential and also the matter then the action is given by

$$S = \frac{1}{2} \int \left[ T \left( \frac{1}{K} + \xi B(\phi) \right) + \partial_\mu \phi \partial^\mu \phi - 2V(\phi) \right] e d^4x + S_m, \quad (5)$$

where  $\xi$  — the coupling constant,

$B(\phi)$  — the nonminimally coupled function,

$V(\phi)$  — the potential of the scalar field  $\phi$ ,

$S_m$  — the matter action,

$K = 8\pi G$ .

A spatially flat Friedmann-Lemaître-Robertson-Walker metric will be used

$$ds^2 = dt^2 - a^2(t) \delta_{ij} dx^i dx^j, \quad (6)$$

where  $i, j = 1, 2, 3$  — spatial indices,

$t$  — the cosmic time,

$x^i$  — co-moving spatial coordinates,  
 $a(t)$  — the scale factor.

Equations of gravitational and scalar fields are derived by varying the action (5) with respect to the tetrad  $e_\mu^A = \text{diag}(1, a(t), a(t), a(t))$ , which corresponds to the chosen metric (6), and with respect to the scalar field

$$3H^2 = K \left( \frac{\dot{\phi}^2}{2} + V(\phi) - 3\xi H^2 B(\phi) + \rho \right), \quad (7)$$

$$2\dot{H} = -K \left( \dot{\phi}^2 + 2\xi H \dot{\phi} B'(\phi) + 2\xi \dot{H} B(\phi) + \rho(1+w) \right), \quad (8)$$

$$\ddot{\phi} + 3H\dot{\phi} + 3\xi H^2 B'(\phi) + V'(\phi) = 0. \quad (9)$$

Here  $H(t) \equiv \frac{\dot{a}}{a}$  is Hubble parameter,  $\rho$  — the energy density of the matter,  $p$  — the pressure of the matter,  $p = w\rho$  — the matter equation of state,  $w \in [-1; 1]$  is a constant, a dot denotes the derivative with respect to time, a prime is the derivative with respect to the scalar field. The torsion scalar is  $T = -6H^2$  in the chosen tetrad.

Equations (7), (8) can be rewritten as

$$3H^2 = K(\rho_\phi + \rho), \quad (10)$$

$$2\dot{H} = -K(p_\phi + \rho_\phi + p + \rho), \quad (11)$$

where

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) - 3\xi H^2 B(\phi), \quad (12)$$

$$p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi) + 2\xi H \dot{\phi} B'(\phi) + \xi(2\dot{H} + 3H^2)B(\phi), \quad (13)$$

In this work we shall consider cosmological models with  $B(\phi) = \phi^N$ ,  $N > 2$  — even;  $V(\phi) = V_0\phi^n$ ,  $n < 0$  — even;  $\xi > 0$ .

## 3 Asymptotic solutions and numerical analysis of their stability

### 3.1 The vacuum case

Several asymptotic power-law regimes (with  $\phi \rightarrow \infty$ ) have been found in cosmological models of teleparallel gravity with nonminimal coupling  $\xi T\phi^N$  for  $N = 2$  and the potential of the form  $V(\phi) = V_0\phi^n$  in our previous papers [22], [23]. As the asymptotic power-law behaviour of Hubble parameter  $H(t)$  and the scalar field  $\phi(t)$  is a typical phenomenon in such models then we could expect an emergence of similar solutions also in the case of  $N > 2$ . Cosmological models with  $N > 2$  have not been studied in articles [22], [23] using expansion-normalized variables as the corresponding dynamical system has zero denominator for  $N \neq 2$ ,  $\phi \rightarrow \infty$ .

We want to check whether or not there is an asymptotic regime of power-law form of  $H(t)$  and  $\phi(t)$  in considered models. Let us substitute the solution  $H(t) = H_0(t - t_0)^\alpha$ ,  $\phi(t) = \phi_0(t - t_0)^{\alpha\beta}$ ,  $\rho = 0$ , where  $\alpha$ ,  $\beta$ ,  $H_0$ ,  $\phi_0$ ,  $t_0$  are constants, to the system (7)-(9), where  $B(\phi) = \phi^N$ ,  $V(\phi) = V_0\phi^n$ .

$$3H_0^2(t - t_0)^{2\alpha} = K \left( \frac{\alpha^2\beta^2}{2}\phi_0^2(t - t_0)^{2\alpha\beta-2} + V_0\phi_0^n(t - t_0)^{n\alpha\beta} - 3\xi H_0^2(t - t_0)^{2\alpha}\phi_0^N(t - t_0)^{N\alpha\beta} \right), \quad (14)$$

$$2\alpha H_0(t-t_0)^{\alpha-1} = -K \left( \alpha^2 \beta^2 \phi_0^2 (t-t_0)^{2\alpha\beta-2} + 2\xi N H_0 (t-t_0)^\alpha \alpha \beta \phi_0^N (t-t_0)^{N\alpha\beta-1} + 2\xi \alpha H_0 (t-t_0)^{\alpha-1} \phi_0^N (t-t_0)^{N\alpha\beta} \right), \quad (15)$$

$$\alpha\beta(\alpha\beta-1)\phi_0(t-t_0)^{\alpha\beta-2} + 3H_0(t-t_0)^\alpha \alpha \beta \phi_0(t-t_0)^{\alpha\beta-1} + 3\xi N H_0^2 (t-t_0)^{2\alpha} \phi_0^{N-1} (t-t_0)^{(N-1)\alpha\beta} + n V_0 \phi_0^{n-1} (t-t_0)^{(n-1)\alpha\beta} = 0. \quad (16)$$

Two cases are considered further: **1).** when the scalar potential is neglected and **2).** when the potential influences the cosmological evolution significantly.

**1).** If  $\alpha < 0$ ,  $\beta < 0$ ,  $N > 2$ ,  $n\alpha\beta < 2\alpha\beta - 2$  and

$$2\alpha\beta - 2 = N\alpha\beta + 2\alpha \Leftrightarrow \alpha\beta = \frac{2(1+\alpha)}{2-N} > 0 \Rightarrow \alpha < -1 \quad (17)$$

then  $2\alpha\beta - 2 < N\alpha\beta + \alpha - 1$  and, neglecting smaller terms for  $t \rightarrow +\infty$ , we receive

$$0 = K \left( \frac{\alpha^2 \beta^2}{2} \phi_0^2 (t-t_0)^{2\alpha\beta-2} - 3\xi H_0^2 \phi_0^N (t-t_0)^{N\alpha\beta+2\alpha} \right), \quad (18)$$

$$0 = -K \left( 2\xi N H_0 \alpha \beta \phi_0^N (t-t_0)^{N\alpha\beta+\alpha-1} + 2\xi \alpha H_0 \phi_0^N (t-t_0)^{N\alpha\beta+\alpha-1} \right), \quad (19)$$

$$\alpha\beta(\alpha\beta-1)\phi_0(t-t_0)^{\alpha\beta-2} + 3\xi N H_0^2 \phi_0^{N-1} (t-t_0)^{(N-1)\alpha\beta+2\alpha} = 0. \quad (20)$$

We obtain from (18), (19), (20) using (17)

$$\beta = -\frac{1}{N}, \quad \alpha = -\frac{2N}{N+2}, \quad \alpha\beta = \frac{2}{N+2}, \quad (21)$$

$$H_0^2 = \frac{2}{3\xi(N+2)^2} \phi_0^{2-N}. \quad (22)$$

Remembering the initial assumption  $n\alpha\beta < 2\alpha\beta - 2$  we get

$$n < -N. \quad (23)$$

**2).** If  $\alpha < 0$ ,  $\beta < 0$ ,  $N > 2$  and  $n\alpha\beta = 2\alpha\beta - 2$ ,  $2\alpha\beta - 2 = N\alpha\beta + 2\alpha$  then

$$\alpha = \frac{n-N}{2-n} < -1, \quad \beta = \frac{2}{n-N}. \quad (24)$$

The system (14)-(16) reduces to next one for this assumptions in the limit  $t \rightarrow +\infty$

$$0 = K \left( \frac{\alpha^2 \beta^2}{2} \phi_0^2 (t-t_0)^{2\alpha\beta-2} + V_0 \phi_0^n (t-t_0)^{n\alpha\beta} - 3\xi H_0^2 \phi_0^N (t-t_0)^{N\alpha\beta+2\alpha} \right), \quad (25)$$

$$0 = -K \left( 2\xi N H_0 \alpha \beta \phi_0^N (t-t_0)^{N\alpha\beta+\alpha-1} + 2\xi \alpha H_0 \phi_0^N (t-t_0)^{N\alpha\beta+\alpha-1} \right), \quad (26)$$

$$\alpha\beta(\alpha\beta-1)\phi_0(t-t_0)^{\alpha\beta-2} + 3\xi N H_0^2 \phi_0^{N-1} (t-t_0)^{(N-1)\alpha\beta+2\alpha} + n V_0 \phi_0^{n-1} (t-t_0)^{(n-1)\alpha\beta} = 0. \quad (27)$$

For

$$n = -N \quad (28)$$

we have

$$\beta = -\frac{1}{N}, \quad \alpha = -\frac{2N}{N+2}, \quad \alpha\beta = \frac{2}{N+2}, \quad (29)$$

and

$$H_0^2 = \frac{2}{3\xi(N+2)^2}\phi_0^{2-N} + \frac{V_0}{3\xi\phi_0^{2N}}. \quad (30)$$

Therefore, summarizing results of **1).**, **2).**, we conclude that the asymptotic solution

$$H(t) = H_0(t-t_0)^{-\frac{2N}{N+2}}, \quad a(t) = a_0 e^{H_0 \frac{N+2}{2-N}(t-t_0)^{\frac{2-N}{N+2}}}, \quad \phi(t) = \phi_0(t-t_0)^{\frac{2}{N+2}} \quad (31)$$

exists for  $N > 2$ ,  $n \leq -N$ ,  $t \rightarrow +\infty$ .

We note that Hubble parameter decreases and tends to zero at the asymptotic solution (31). The scale factor  $a(t)$  increases slowly and approaches the constant  $a_0$  in the limit  $t \rightarrow +\infty$ .

The first order system of differential equation (32) is obtained from initial equations (7)-(9) for  $\rho = 0$  and it is integrated numerically.

$$\begin{aligned} \dot{H} &= -\frac{K(\Phi^2 + 2\xi H \Phi B'(\phi))}{2(1 + K\xi B(\phi))}, \\ \dot{\phi} &= \Phi, \\ \dot{\Phi} &= -3H\Phi - 3\xi H^2 B'(\phi) - V'(\phi), \\ \dot{a} &= aH, \end{aligned} \quad (32)$$

where the equality  $H^2 = (\Phi^2 + 2V(\phi))/(6(1 + K\xi B(\phi)))$  is checked at each step of the numerical integration.

We construct following auxiliary quantities

$$D(t) = \frac{d(\ln(\phi))}{d(\ln(t))} = \frac{t\dot{\phi}}{\phi}, \quad E(t) = \frac{d(\ln(\dot{\phi}))}{d(\ln(t))} = \frac{t\ddot{\phi}}{\dot{\phi}}, \quad F(t) = \frac{d(\ln(H))}{d(\ln(t))} = \frac{t\dot{H}}{H}, \quad (33)$$

which are useful in the graphic representation since they approach constants if  $\phi(t)$ ,  $\dot{\phi}(t)$ ,  $H(t)$  draw near to a power-law behaviour. The Fig. 1 demonstrates functions  $D(t)$ ,  $E(t)$ ,  $F(t)$  obtained by numerical integration for fixed parameters  $N \leq -n$ . We see that  $D(t) \rightarrow \frac{2}{N+2}$ ,  $E(t) \rightarrow -\frac{N}{N+2}$ ,  $F(t) \rightarrow -\frac{2N}{N+2}$  for the asymptotic regime (31),  $t \rightarrow +\infty$ . Therefore, power indices (29) coincide with those which are found by applying the numerical investigation.

Moreover, the late-time tendency of auxiliary functions (33) to constants  $\frac{2}{N+2}$ ,  $\frac{N}{N+2}$ ,  $\frac{2N}{N+2}$  are got numerically for the set points of the plane  $(\phi(0), \dot{\phi}(0))$  and for various values of parameters  $N$ ,  $n$ ,  $\xi$ . Consequently, the found vacuum asymptotic solution is stable in time for some area of initial conditions.

Dependences  $a(t)$ ,  $H(t)$ ,  $\phi(t)$  are plotted in Fig. 2, 3, 4 for the same initial data and parameters as in Fig. 1.

### 3.2 The matter case

If the matter is added ( $\rho \neq 0$ ) to considered model then the power-law solution (31) does not exist in the limit  $t \rightarrow +\infty$ . In this case the matter term dominates in comparison with other items in equations of motion (7)-(8). Namely, it follows from the continuity equation  $\dot{\rho} + 3H(1+w)\rho = 0$  that

$$\rho(t) = \rho_0 e^{\frac{-3(1+w)H_0}{\alpha+1}(t-t_0)^{\alpha+1}} \rightarrow \rho_0, \quad (34)$$

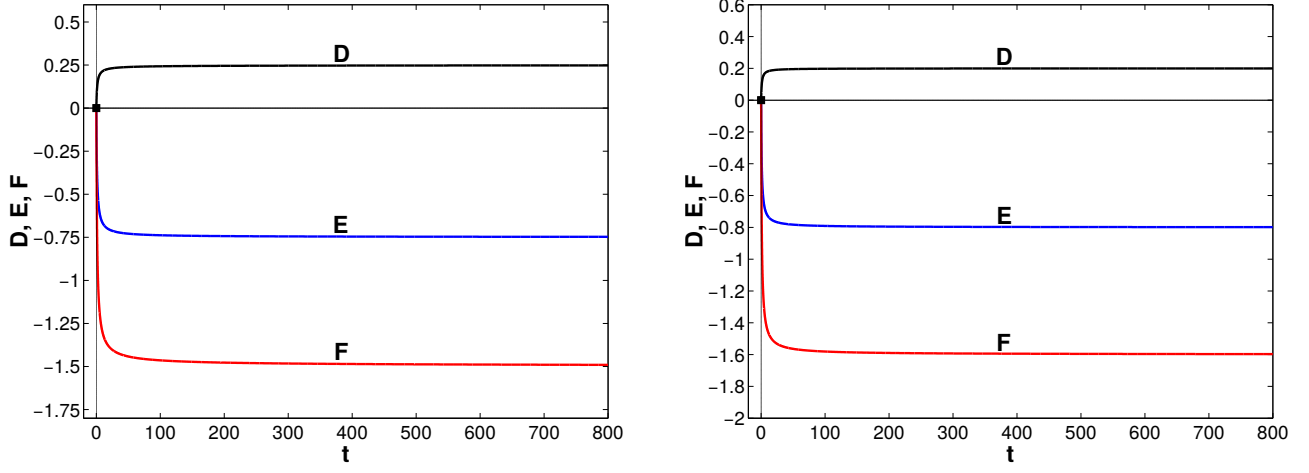


Fig. 1. Evolution of quantities  $D(t)$ ,  $E(t)$ ,  $F(t)$  for initial data:  $\phi(0) = 2$ ,  $\dot{\phi}(0) = 0.4$ ,  $a(0) = 1$ . Parameters are chosen  $N = -n = 6$  for the left plot and  $N = 8$ ,  $n = -10$  for the right one. Other parameters:  $\xi = 1$ ,  $V_0 = 1$ ,  $K = 1$ . Black squares are initial values  $D(0)$ ,  $E(0)$ ,  $F(0)$ .

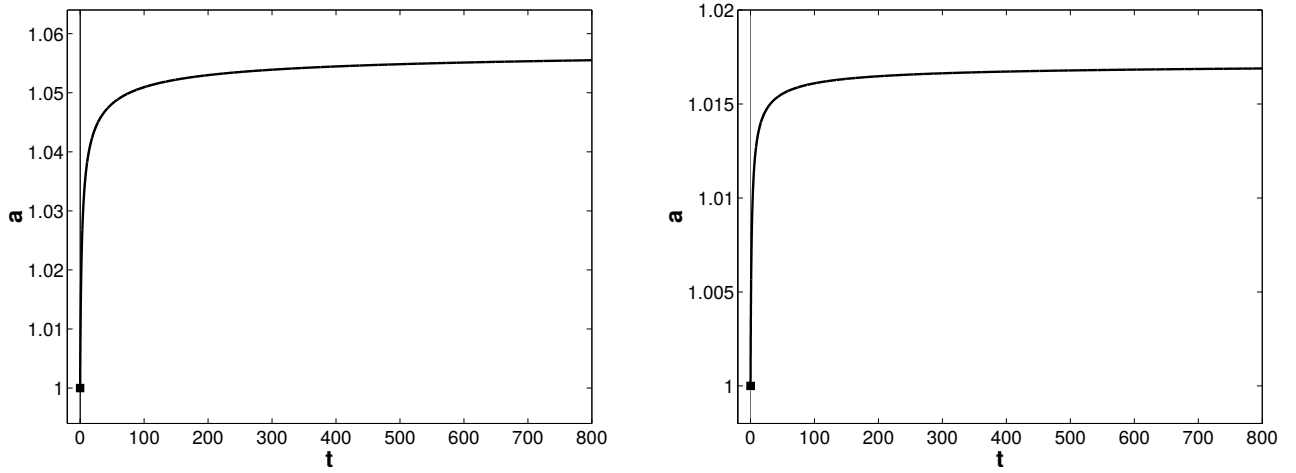


Fig. 2. Evolution of  $a(t)$  for initial data:  $\phi(0) = 2$ ,  $\dot{\phi}(0) = 0.4$ ,  $a(0) = 1$ . Parameters are chosen  $N = -n = 6$  for the left graph and  $N = 8$ ,  $n = -10$  for the right one. Other parameters:  $\xi = 1$ ,  $V_0 = 1$ ,  $K = 1$ . Black squares are initial values  $a(0)$ .

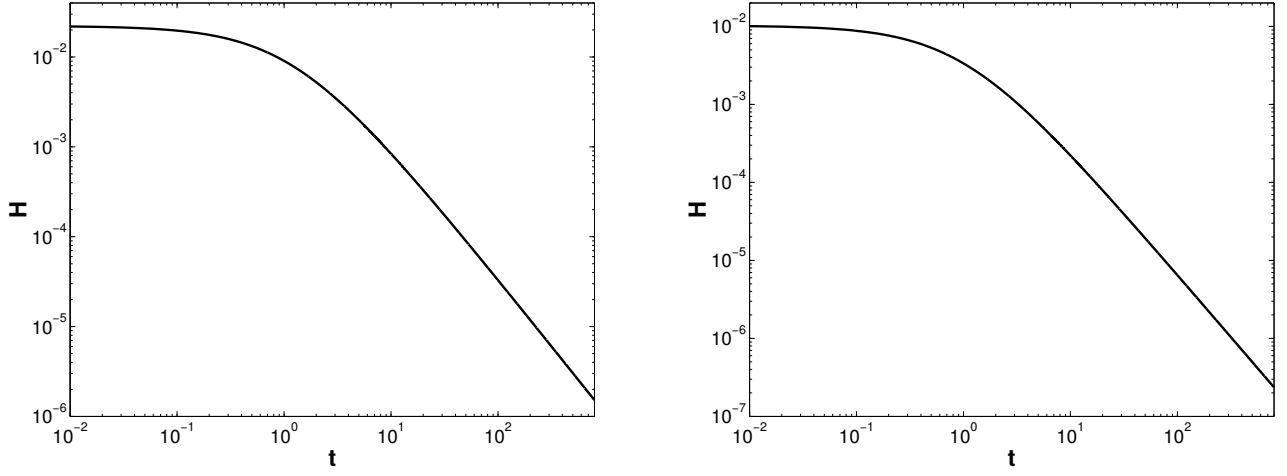


Fig. 3. Evolution of  $H(t)$  are plotted for initial data:  $\phi(0) = 2$ ,  $\dot{\phi}(0) = 0.4$ ,  $a(0) = 1$ . The starting value of Hubble parameter  $H(0)$  is calculated using the constraint equation (7). Parameters are chosen  $N = -n = 6$  for the left graph and  $N = 8$ ,  $n = -10$  for the right one. Other parameters:  $\xi = 1$ ,  $V_0 = 1$ ,  $K = 1$ .

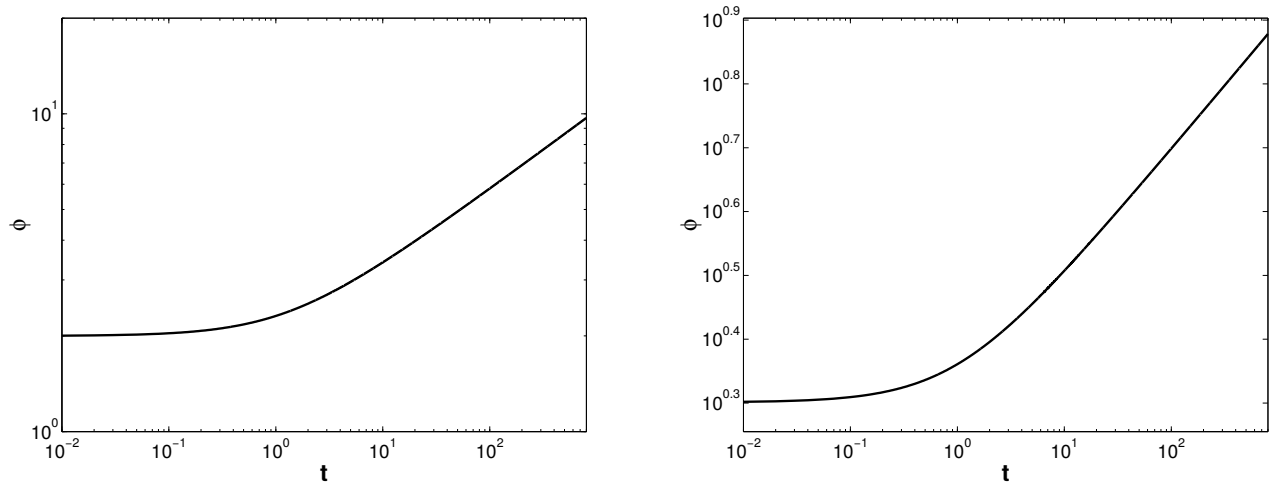


Fig. 4. Evolution of  $\phi(t)$  are plotted for initial data:  $\phi(0) = 2$ ,  $\dot{\phi}(0) = 0.4$ ,  $a(0) = 1$ . Parameters are chosen  $N = -n = 6$  for the left plot and  $N = 8$ ,  $n = -10$  for the right one. Other parameters:  $\xi = 1$ ,  $V_0 = 1$ ,  $K = 1$ .

while

$$\begin{aligned}\frac{\dot{\phi}^2}{2} &= \frac{\alpha^2 \beta^2}{2} \phi_0^2 (t - t_0)^{2\alpha\beta-2} \rightarrow 0, \\ -3\xi H^2 \phi^N &= -3\xi H_0^2 \phi_0^N (t - t_0)^{N\alpha\beta+2\alpha} \rightarrow 0,\end{aligned}\quad (35)$$

$$V(\phi) = V_0 \phi_0^n (t - t_0)^{n\alpha\beta} \rightarrow 0$$

for  $\alpha < -1$ ,  $\beta < 0$ ,  $\alpha\beta < 1$ ,  $N > 2$ ,  $n \leq -N$ ,  $t \rightarrow +\infty$ .

However, if the cosmological evolution begins from very small value of  $\rho(0)$  then the quasistatic phase with  $a \approx \text{const}$  can be realized during some time interval (so called “loitering universe”, this scenario in GR is described, for example, in [24]). The quasistatic stage finishes when the energy density of the matter  $\rho(t)$  begins to prevail over other terms in (7)-(8).

Actually, the cosmological evolution in considered models is more difficult for small  $\rho(0)$ . Using the numerical integration of the system of differential equations (36)

$$\begin{aligned}\dot{H} &= -\frac{K[\Phi^2 + 2\xi H \Phi B'(\phi) + (1+w)(3H^2(1/K + \xi B(\phi)) - \Phi^2/2 - V(\phi))]}{2(1 + K\xi B(\phi))}, \\ \dot{\phi} &= \Phi, \\ \dot{\Phi} &= -3H\Phi - 3\xi H^2 B'(\phi) - V'(\phi), \\ \dot{a} &= aH,\end{aligned}\quad (36)$$

where  $\rho = 3H^2(1/K + \xi B(\phi)) - \Phi^2/2 - V(\phi)$  was substituted, we show that the scale factor behaves like the step function at early time for  $\rho(0) \lesssim 10^{-3}$  and we have at least two temporary quasistatic stages (see Fig. 5). It is worth emphasizing that the existence of more than one phases with  $a \approx \text{const}$  is not obvious from the analytical form of equations of motion and those have been found only by numerical methods. Such behaviour of the scale factor is provided by scalar field oscillations.

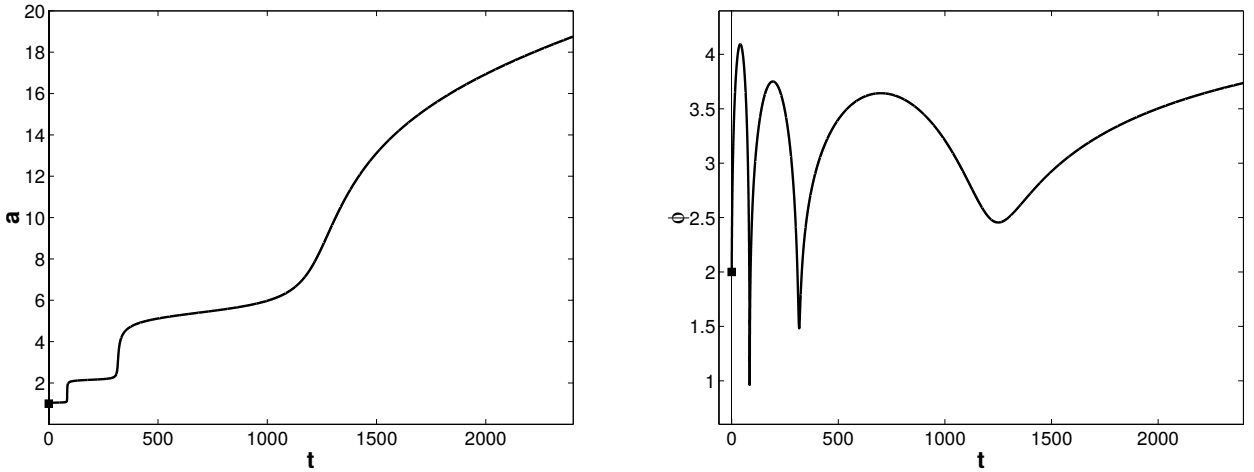


Fig. 5. Evolution of  $a(t)$ ,  $\phi(t)$  for initial data:  $\phi(0) = 2$ ,  $\dot{\phi}(0) = 0.4$ ,  $H(0) = 0.02226$ ,  $a(0) = 1$ . Parameters are chosen  $N = 6$ ,  $n = -6$ ,  $\xi = 1$ ,  $V_0 = 1$ ,  $w = 0$ ,  $K = 1$ .

Now we check the existence of the asymptotic power-law regime  $a(t) = a_0(t - t_0)^A$ ,  $(H(t) = A(t - t_0)^{-1})$ ,  $\phi(t) = \phi_0(t - t_0)^{AL}$ ,  $\rho(t) = \rho_0(t - t_0)^{-3(1+w)A}$ , where  $A$ ,  $L$ ,  $a_0$ ,  $\phi_0$ ,  $\rho_0$ ,  $t_0$  are constants. Substituting it to the initial system (7)-(9) for  $B(\phi) = \phi^N$ ,  $V(\phi) = V_0 \phi^n$  we get

$$\begin{aligned}3A^2(t - t_0)^{-2} &= K \left( \frac{A^2 L^2}{2} \phi_0^2 (t - t_0)^{2AL-2} + V_0 \phi_0^n (t - t_0)^{nAL} - \right. \\ &\quad \left. - 3\xi A^2(t - t_0)^{-2} \phi_0^N (t - t_0)^{NAL} + \rho_0(t - t_0)^{-3(1+w)A} \right),\end{aligned}\quad (37)$$



$$-2A(t-t_0)^{-2} = -K \left( A^2 L^2 \phi_0^2 (t-t_0)^{2AL-2} + 2\xi N A (t-t_0)^{-1} AL \phi_0^N (t-t_0)^{NAL-1} - 2\xi A (t-t_0)^{-2} \phi_0^N (t-t_0)^{NAL} + (1+w)\rho_0 (t-t_0)^{-3(1+w)A} \right), \quad (38)$$

$$AL(AL-1)\phi_0(t-t_0)^{AL-2} + 3A(t-t_0)^{-1}AL\phi_0(t-t_0)^{AL-1} + 3\xi N A^2(t-t_0)^{-2}\phi_0^{N-1}(t-t_0)^{(N-1)AL} + nV_0\phi_0^{n-1}(t-t_0)^{(n-1)AL} = 0. \quad (39)$$

If  $A > 0$ ,  $L > 0$ ,  $N > 2$ ,  $n < 0$  and  $nAL = NAL - 2$ ,  $NAL - 2 = -3(1+w)A$  then

$$A = \frac{2n}{3(1+w)(n-N)}, \quad L = -\frac{3(1+w)}{n}. \quad (40)$$

Using that we keep only dominating terms in the limit  $t \rightarrow +\infty$  from the system (37)-(39)

$$0 = K \left( V_0 \phi_0^n (t-t_0)^{nAL} - 3\xi A^2 \phi_0^N (t-t_0)^{NAL-2} + \rho_0 (t-t_0)^{-3(1+w)A} \right), \quad (41)$$

$$0 = -K \left( 2\xi N A^2 L \phi_0^N (t-t_0)^{NAL-2} - 2\xi A \phi_0^N (t-t_0)^{NAL-2} + \rho_0 (1+w)(t-t_0)^{-3(1+w)A} \right), \quad (42)$$

$$3\xi N A^2 \phi_0^{N-1} (t-t_0)^{(N-1)AL-2} + nV_0 \phi_0^{n-1} (t-t_0)^{(n-1)AL} = 0 \quad (43)$$

and find

$$\begin{aligned} \rho_0 &= \frac{4n\xi\phi_0^N(N+n)}{3(N-n)^2(1+w)^2} > 0 \quad \Rightarrow \quad n < -N, \\ V_0 &= -\frac{4n\xi N \phi_0^{N-n}}{3(N-n)^2(1+w)^2}. \end{aligned} \quad (44)$$

We obtain the asymptotic solution

$$a(t) = a_0(t-t_0)^{\frac{2n}{3(1+w)(n-N)}}, \quad \phi(t) = \phi_0(t-t_0)^{\frac{2}{N-n}}, \quad \rho(t) = \rho_0(t-t_0)^{\frac{2n}{N-n}}, \quad (45)$$

which exists for  $N > 2$ ,  $n < -N$ ,  $t \rightarrow +\infty$ .

The scale factor and the scalar field increase at the asymptotic solution (45) while Hubble parameter and the energy density of the matter decrease to zero in the limit  $t \rightarrow +\infty$ .

Taking account (10), (11) we obtain from equations (41), (42)  $\rho_\phi(t) = -\rho(t)$ ,  $p_\phi(t) = -p(t)$  and, therefore,  $w_\phi = \frac{p_\phi}{\rho_\phi} = w = \frac{p}{\rho}$ . This is the property of a tracker solution. An analogous regime has been found earlier for  $N = 2$  in [23].

Auxiliary functions are introduced the same way as in the previous Subsection

$$M(t) = \frac{d(\ln(a))}{d(\ln(t))} = tH, \quad D(t) = \frac{d(\ln(\phi))}{d(\ln(t))} = \frac{t\dot{\phi}}{\phi}, \quad R(t) = \frac{d(\ln(\rho))}{d(\ln(t))} = -3(1+w)tH. \quad (46)$$

Such quantities have the simple form of a constant in corresponding plots when the cosmological evolution passes through the power-law stage. Time dependences of these functions are plotted in Fig. 6 (left). These functions approach power indices  $M(t) \rightarrow \frac{2n}{3(1+w)(n-N)}$ ,  $D(t) \rightarrow \frac{2}{N-n}$ ,  $R(t) \rightarrow \frac{2n}{N-n}$  for the asymptotic regime (45),  $t \rightarrow +\infty$ .

The numerical integration has been carried out for the set points of the space  $(\phi(0), \dot{\phi}(0), H(0))$  and for various parameters  $N$ ,  $n$ ,  $\xi$ . We have received that quantities  $M(t)$ ,  $D(t)$ ,  $R(t)$  tend to constants  $\frac{2n}{3(1+w)(n-N)}$ ,  $\frac{2}{N-n}$ ,  $\frac{2n}{N-n}$  correspondingly. Therefore, the region of initial data exists for which the power-law solution (45) is stable in time.

The behaviour of the scale factor, the scalar field and the energy density of the matter are shown in Fig. 6 (right), Fig. 7, where initial conditions and parameters are those as in Fig. 6.

Finally, we note that asymptotic power-law solutions have not been revealed neither numerically, nor analytically in the matter case for  $N = -n$ . This case needs further analysis.

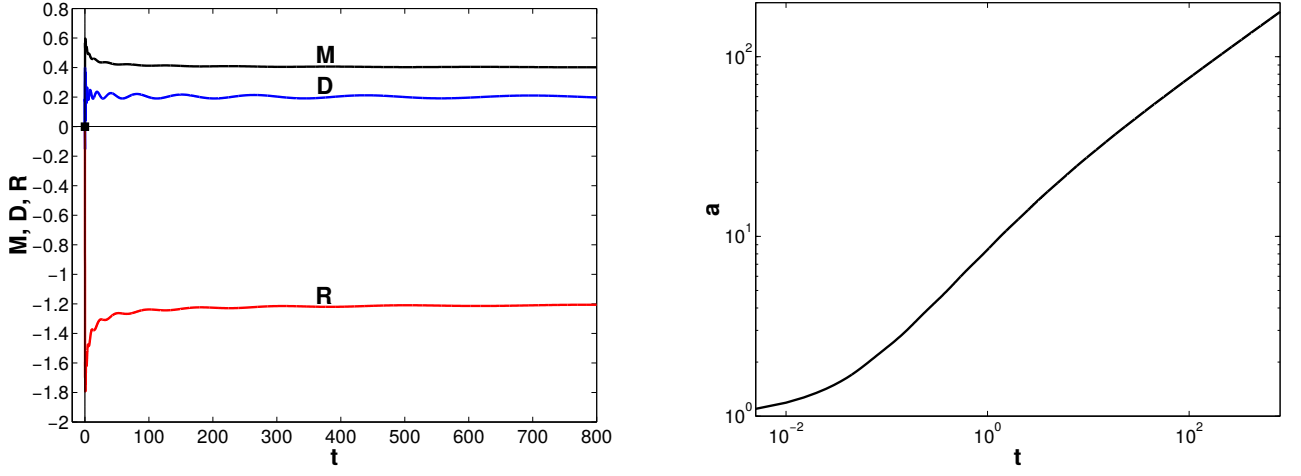


Fig. 6. Evolution of quantities  $M(t)$ ,  $D(t)$ ,  $R(t)$  in the left graph and the scalar factor  $a(t)$  in the right one for initial data:  $\phi(0) = 0.4$ ,  $\dot{\phi}(0) = 2$ ,  $H(0) = 20$ ,  $a(0) = 1$ . Parameters are chosen  $N = 4$ ,  $n = -6$ ,  $\xi = 4$ ,  $V_0 = 1$ ,  $w = 0$ ,  $K = 1$ . Black squares are initial values  $M(0)$ ,  $D(0)$ ,  $R(0)$ .

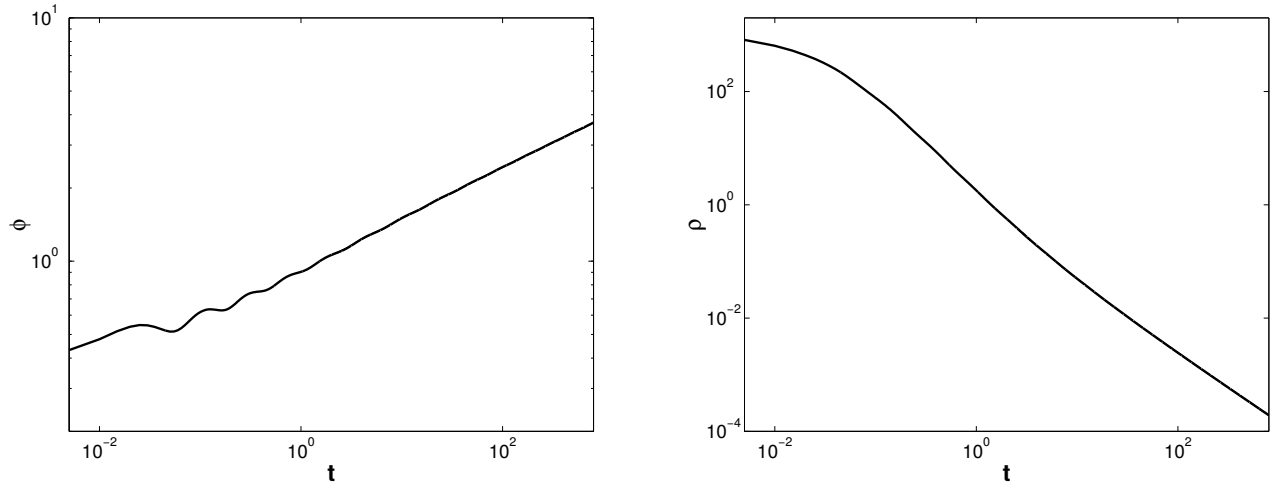


Fig. 7. Evolution of  $\phi(t)$ ,  $\rho(t)$  for initial data:  $\phi(0) = 0.4$ ,  $\dot{\phi}(0) = 2$ ,  $H(0) = 20$ ,  $a(0) = 1$ . The starting value of the energy density  $\rho(0)$  of the matter is calculated using the constraint equation (7). Parameters are chosen  $N = 4$ ,  $n = -6$ ,  $\xi = 4$ ,  $V_0 = 1$ ,  $w = 0$ ,  $K = 1$ .

## 4 Conclusion

In the present paper the evolution of the late Universe has been studied in models of teleparallel gravity with nonminimal coupling  $\xi T\phi^N$ ,  $N > 2$ ,  $\xi > 0$ , the field potential  $V(\phi) = V_0\phi^n$ ,  $n < 0$  and a perfect fluid with the equation of state  $p = w\rho$ . We find analytically new asymptotic solutions in the vacuum case and in the presence of a perfect fluid for  $t \rightarrow +\infty$ .

The vacuum asymptotic regime  $H(t) = H_0(t - t_0)^{-\frac{2N}{N+2}} \rightarrow 0$  ( $a(t) \rightarrow \text{const}$ ),  $\phi(t) = \phi_0(t - t_0)^{\frac{2}{N+2}} \rightarrow \infty$  exists only for  $N > 2$ ,  $n \leq -N$ . The found asymptotic solution exists when the term with the nonminimal coupling, the kinetic energy of the scalar field for  $n < -N$  and the potential for  $n = -N$  dominate. We note that the deceleration parameter  $q \equiv -\frac{\ddot{a}}{aH^2}$  corresponding to the obtained solution is  $q = -1 + \frac{2N}{H_0(N+2)}(t - t_0)^{\frac{N-2}{N+2}} \rightarrow +\infty$ ,  $t \rightarrow +\infty$ . Therefore, the Universe expands with the deceleration at late time and approaches asymptotically the static state.

We have shown numerically (see Fig. 1, 2, 3, 4) that the found vacuum solution is stable in time for some region of initial data. Therefore, Einstein's plan to receive the stable static Universe in GR is realized in the considered model of teleparallel gravity with the nonminimally coupled scalar field. This result is rather surprising as not so many stable stationary cosmological models are known.

Any perfect fluid ( $\rho \neq 0$ ) destroys the vacuum asymptotic solution for  $t \rightarrow +\infty$  as in this case the energy density of the matter prevails over other components of the field equations. However, if the cosmological evolution starts from small  $\rho$  then it passes through several transient quasistatic stages with  $a \approx \text{const}$  similarly to "loitering universe" in GR.

In the matter case other stable asymptotic regime exists for  $N > 2$ ,  $n < -N$ :  $a(t) = a_0(t - t_0)^{\frac{2n}{3(1+w)(n-N)}} \rightarrow +\infty$ ,  $\phi(t) = \phi_0(t - t_0)^{\frac{2}{N-n}} \rightarrow \infty$ ,  $\rho(t) = \rho_0(t - t_0)^{\frac{2n}{N-n}} \rightarrow 0$ ,  $t \rightarrow +\infty$ . The corresponding deceleration parameter is  $q = -1 + \frac{3(1+w)(n-N)}{2n} > -1$ . Therefore, an acceleration expansion is possible for  $w \leq -\frac{2}{3}$ . The nonminimally coupled term, the potential energy and the matter energy density dominate at this asymptotic solution. It is a tracker as  $w_\phi = w$ . Such behaviour of the scalar field can be suitable for a description of radiation- and matter-dominated stages of the Universe.

The numerical integration confirms the existence of the tracker solution for  $t \rightarrow +\infty$ , which is stable in time for some region of initial conditions. Plots in Fig. 6, 7 (for fixed  $N$  and  $n$ ) show that functions  $a(t)$ ,  $\phi(t)$  and  $\rho(t)$  approach a power-law behaviour and corresponding power indices coincide with those found analytically.

We write down possible final of cosmological evolution in considered models with  $N > 2$ ,  $n < 0$ :

### 1. $\rho = 0$ .

1).  $-N < n < 0$  — stable de Sitter solution (see [22], [23])  $H = H_0$ ,  $\phi = \phi_0$ ,  $q = -1$ ,

2).  $n \leq -N$  — stable asymptotic solution  $H(t) = \frac{H_0}{(t-t_0)^{\frac{2N}{N+2}}}$ ,  $\phi = \phi_0(t - t_0)^{\frac{2}{N+2}}$ ,  
 $q = -1 + \frac{2N}{H_0(N+2)}(t - t_0)^{\frac{N-2}{N+2}} \rightarrow +\infty$ ,  $t \rightarrow +\infty$ .

### 2. $\rho \neq 0$ .

1).  $-N < n < 0$  — stable de Sitter solution (see [23])  $H = H_0$ ,  $\phi = \phi_0$ ,  $q = -1$ ,

- 2).  $n < -N$  — stable asymptotic solution  $a(t) = a_0(t - t_0)^{\frac{2n}{3(1+w)(n-N)}}$ ,  $\phi = \phi_0(t - t_0)^{\frac{2}{N-n}}$ ,  
 $\rho = \rho_0(t - t_0)^{\frac{2n}{N-n}}$ ,  $q = -1 + \frac{3(1+w)(n-N)}{2n} = \text{const} > -1$ .

Therefore, we see that strongly decreasing potentials ( $n \leq -N$ ) lead to the late-time behaviour of cosmological quantities differing from one at the cosmic acceleration stage of the real Universe. However, models with decreasing potential and a perfect fluid might be interesting due to existence the tracker solution, which can be used for constructing realistic cosmological models.

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